# Linear-Quadratic Guidance Law for Dual Control of Homing Missiles

D. G. Hull,* J. L. Speyer,† and D. B. Burris‡
*University of Texas, at Austin, Austin, Texas*

**A linear-quadratic guidance law, called the dual-control guidance law, is proposed for enhancing the information content of the intercept trajectory of a homing missile. The feasibility of this guidance law is established by comparing its tracking performance with that of a linear-quadratic guidance law based on miss distance only.**

## Nomenclature

| | |
|---|---|
| $a$ | = acceleration, ft/s$^2$ |
| $A,B$ | = constant matrices in state equation |
| $c$ | = constant weight in linear-quadratic guidance law, s$^{-3}$ |
| $C$ | = gain |
| $H$ | = measurement matrix |
| $I$ | = observability Gramian |
| $J$ | = performance index |
| $N$ | = navigation ratio |
| $t$ | = time, s |
| $u,v,w$ | = components of relative velocity, ft/s |
| $U$ | = control vector |
| $V$ | = positive-definite, diagonal weighting matrix (constant) |
| $w$ | = constant weight in dual-control guidance law, s$^{-4}$ |
| $W$ | = positive-definite, diagonal weighting matrix (time varying) |
| $x,y,z$ | = components of relative position, ft |
| $X$ | = state vector |
| $Y$ | = matrix in gain vector |
| $\Delta t$ | = $t - t_0$ |
| $\theta$ | = azimuth angle |
| $\lambda$ | = reciprocal of the target maneuver time constant, s$^{-1}$ |
| $\Lambda$ | = matrix in gain vector |
| $\phi$ | = elevation angle |
| $\Phi$ | = transition matrix |

*Subscripts*

| | |
|---|---|
| $f$ | = final point |
| go | = to go |
| $M$ | = missile |
| 0 | = initial point |
| $T$ | = target |
| $x,y,z$ | = components in inertial space |
| 3 | = 3 × 3 matrix |

## I. Introduction

IN the analyses of missile guidance systems, it is usually assumed that the navigation filter and the guidance law can be designed separately. Although it is true that the filter structure does not depend on the control law,[1] the control law does depend on the information received from the filter, and the control law affects the performance of the filter. Hence, it is important to design the guidance law for a homing missile with the performance of the filter in mind. This dual role of the guidance law intercept and estimation enhancement is referred to as dual control.

Previous research in this area has been concerned with the understanding of the problem and has not led to a mechanizable dual-control guidance law. Early studies have considered guidance laws in cascade with an extended Kalman filter. In Refs. 2 and 3, numerical optimization is used to maximize the information content of an intercept trajectory of a realistic model of a homing missile. The performance index is taken to be the trace of the Fisher information matrix (observability Gramian), and the resulting trajectory indicates that the line of sight between the missile and the target should be kept in motion to increase filter observability. In Refs. 4 and 5, simple engagement dynamics are employed to compute maximum-information paths; however, the structure of the optimal control problem is still too complex to yield an analytical solution. Finally, in Ref. 6, the problem is formulated as the minimization of the covariance matrix at the final point. The performance index is the smallest eigenvalue of the information matrix or observability Gramian. Again, the structure of the problem is too complex to result in a simple solution.

Recently, it has been discovered that the nonlinear measurement functions associated with angle-only measurements can be expressed in a linear form.[7] This has led to the development in Ref. 8 of a new filter structure called the modified-gain extended Kalman filter. Application of this filter along a maximum-information intercept path is discussed in Ref. 9.

In this paper, a linear-quadratic guidance law (LQGL) with estimation enhancement features associated with the modified-gain extended Kalman filter is discussed. The performance index has been proposed by Speyer, investigated in two-dimensions with Riccati equations in Ref. 10, and studied in more detail in Ref. 11. The present paper is a summary of these results.

In Sec. II, the standard linear-quadratic guidance law is reviewed. Next, an information-enhancing term is motivated in Sec. III, and, in Sec. IV, a linear-quadratic guidance law that accomplishes intercept and information enhancement, called the dual-control guidance law (DCGL), is developed. Then, tracking results along paths generated by the two guidance laws are presented in Sec. V. Finally, some discussion and conclusions are presented in Sec. VI. The Appendix contains a definition of the linear-quadratic optimal control problem and the equations for the dual-control guidance law.

## II. Linear-Quadratic Guidance Law

A commonly used guidance law for homing missiles is called proportional navigation. It is based on the principle of driving the line-of-sight angle rate to zero in order to keep the missile and target in a homing triangle. Theoretically, this

Received Dec. 3, 1987; revision received May 13, 1988. 

*M. J. Thompson Regents Professor, Department of Aerospace Engineering and Engineering Mechanics. Associate Fellow AIAA.

†Harry H. Power Professor in Engineering, Department of Aerospace Engineering and Engineering Mechanics. Fellow AIAA.

‡Graduate Research Assistant, Department of Aerospace Engineering and Engineering Mechanics; currently, Senior Aerospace Engineer, LTV, Dallas, TX.

guidance law is valid only for engagement geometries in which the missile does not deviate very much from the homing triangle. Also, when the missile and the target are approximately in the homing triangle, the state of the engagement has very low observability.

A more recently developed guidance law, which contains proportional navigation as a particular case, called the linear-quadratic guidance law, is based on the minimization of the performance index[12]

$$J = \frac{c}{2}(x_f^2 + y_f^2 + z_f^2) + \frac{1}{2}\int_{t_0}^{t_f} (a_{M_x}^2 + a_{M_y}^2 + a_{M_z}^2)\,\mathrm{d}t \tag{1}$$

subject to the linear differential constraints

$$\dot{x} = u \tag{2}$$

$$\dot{y} = v \tag{3}$$

$$\dot{z} = w \tag{4}$$

$$\dot{u} = a_{T_x} - a_{M_x} \tag{5}$$

$$\dot{v} = a_{T_y} - a_{M_y} \tag{6}$$

$$\dot{w} = a_{T_z} - a_{M_z} \tag{7}$$

$$\dot{a}_{T_x} = -\lambda a_{T_x} \tag{8}$$

$$\dot{a}_{T_y} = -\lambda a_{T_y} \tag{9}$$

$$\dot{a}_{T_z} = -\lambda a_{T_z} \tag{10}$$

subject to the prescribed initial conditions

$$t_0, x_0, y_0, z_0, u_0, v_0, w_0, a_{T_{x0}}, a_{T_{y0}}, a_{T_{z0}} \quad \text{given} \tag{11}$$

and subject to the prescribed final condition

$$t_f \quad \text{given} \tag{12}$$

In these equations, $x$, $y$, $z$, $u$, $v$, and $w$ are the components of the relative position vector and the relative velocity vector in inertial coordinates (see Fig. 1). Also, $a_{T_x}$, $a_{T_y}$, $a_{T_z}$, $a_{M_x}$, $a_{M_y}$, and $a_{M_z}$ are the inertial components of the target and missile acceleration vectors. The quantities $c$ and $\lambda$ are discussed in subsequent paragraphs.

The purpose of the performance index [Eq. (1)] is to minimize the miss distance with a penalty on the magnitude of the missile acceleration. The weighting factor $c$ is a design parameter whose value is chosen to balance miss distance and missile acceleration.

The differential equations (2–10) are the deterministic forms of the stochastic differential equations used in the navigation filter to estimate the engagement states from measurements made by the missile.[13] Equations (2–7) are the kinematic equations of relative position and relative velocity. In the filter, Eqs. (8–10) with additive white noise represent first-order Gauss-Markov processes and are used to keep the filter tracking target acceleration changes. The quantity $\lambda$ is the reciprocal of the target acceleration time constant and is a design parameter whose value can be chosen to tune filter performance. In the filter, the missile acceleration components are assumed to be measured, although in the development of the guidance law, the missile acceleration components are the control variables.

The prescribed initial conditions in Eq. (11) are assumed to be available from the navigation filter. On the other hand, the prescribed final condition in Eq. (12) presents a problem because the final time is not actually known. Hence, some algorithm for predicting final time is needed, and the formulas currently being proposed are not acceptable for all engagement geometries.

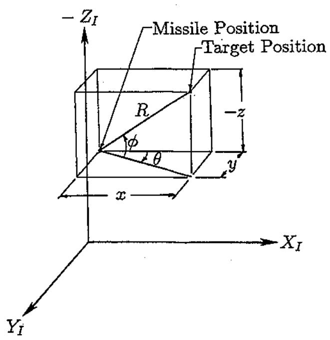


**Fig. 1 Intercept geometry and measurement angles.**

A disadvantage of the problem formulation represented by Eqs. (1–12) is that the missile actually cannot control all three missile acceleration components. Hence, in the implementation of the guidance law, the commanded missile acceleration is projected onto the missile axes; the longitudinal component is disregarded; and the lateral and normal components are used to create control commands.

An advantage of this problem formulation is that the optimal control problem splits into three separate problems—one along each axis. Hence, the optimal control problem actually solved is to find the missile acceleration history $a_{M_y}(t)$, that minimizes the performance index

$$J = \frac{c}{2}y_f^2 + \frac{1}{2}\int_{t_0}^{t_f} a_{M_y}^2\,\mathrm{d}t \tag{13}$$

subject to the differential constraints

$$\dot{y} = v \tag{14}$$

$$\dot{v} = a_{T_y} - a_{M_y} \tag{15}$$

$$\dot{a}_{T_y} = -\lambda a_{T_y} \tag{16}$$

subject to the prescribed initial conditions

$$t_0, y_0, v_0, a_{T_{y0}} \quad \text{given} \tag{17}$$

and subject to the prescribed final condition, Eq. (12).

Structurally, this is a linear-quadratic problem (see the Appendix), and its solution is given by

$$a_{M_y} = C_1 y + C_2 v + C_3 a_{T_y} \tag{18}$$

where the gains are defined as

$$C_1 = \frac{N}{t_{go}^2} \tag{19}$$

$$C_2 = \frac{N}{t_{go}} \tag{20}$$

$$C_3 = \frac{N[\exp(-\lambda t_{go}) + \lambda t_{go} - 1]}{(\lambda t_{go})^2} \tag{21}$$

where $N$ is the navigation ratio

$$N = \frac{3ct_{go}^3}{ct_{go}^3 + 3} \tag{22}$$

and the time-to-go is

$$t_{go} = t_f - t_0 \tag{23}$$

If the weight $c$ is taken to be large so that miss distance is emphasized at the expense of missile acceleration, the navigation ratio, Eq. (22), becomes $N = 3$. If $\lambda$ is made large, then

$C_3 \to 0$. Finally, if small angles are assumed, the LQGL Eq. (18), reduces to proportional navigation.[12]

If a bank-to-turn missile in a six-degree-of-freedom simulation is guided by the LQGL with time-to-go being calculated as range divided by closing speed, the trajectory has the form shown in Fig. 2, where the target is performing a 9 $g$ accelerated maneuver. See Table 1 for the values of the constants used in the calculations. The final time is 3.4 s and the miss distance is 0.70 ft. It is noted that the geometry toward the end of the engagement has the appearance of the homing triangle. As a consequence, a navigation filter would be unable to produce accurate estimates of the state. In other words, the system has low observability, and if the target were to maneuver near the end of the engagement, the missile would not be able to track it. A consequence of this discussion is that it would be desirable to include something in the quadratic performance index that has the effect of making the system more observable. The additional term is motivated in the next section.

## III. Observability of the Navigation Filter

To develop a filter that estimates the engagement states from measurements made by the seeker, mathematical models for the system dynamics and measurements are needed. The deterministic form of the dynamical model is given by Eqs. (2–10). The measurements are assumed to be the azimuth angle $\theta$ and the elevation angle $\phi$ of the current line of sight relative to the initial line of sight (Fig. 1). These angles can be related to the state as follows:

$$\theta = \tan^{-1}(y/x) \tag{24}$$

$$\phi = \tan^{-1}\frac{-z}{(x^2+y^2)^{1/2}} \tag{25}$$

Unfortunately, although the dynamic model is linear, the measurement model is nonlinear and must be linearized to develop a filter such as the extended Kalman filter. The application of the extended Kalman filter to homing missile guidance using the dynamic model [Eqs. (2–10)] and the measurement model [Eqs. (24) and (25)] is presented in Ref. 13. In general, the extended Kalman filter does not work well, partially because of process noise and partially because of low observability due to the guidance law.

Recently, it has been discovered that the angle-only measurements [Eqs. (24) and (25)] can be expressed in a linear form[7] so that the linearization is no longer necessary. The new forms of the measurement equations are as follows:

$$(\sin\theta)x - (\cos\theta)y = 0 \tag{26}$$

$$(\sin\phi\,\cos\theta)x + (\sin\phi\,\sin\theta)y + (\cos\phi)z = 0 \tag{27}$$

which can be verified by substitution of Eqs. (24) and (25). This new measurement model has motivated the development of a filter specifically for the homing missile problem. It is called the modified-gain extended Kalman filter,[8] and its use is compared with that of the extended Kalman filter in Ref. 9. It has been shown in Refs. 7 and 8 that the pseudolinear filter (PLF) produces biased estimates. Since both the filter gains and the residual process are functions of the current noisy measurement, their product in the filter induces correlation that results in the estimate bias. To avoid this problem, it is suggested that the gains be computed only as functions of the a priori measurement history. For the modified-gain extended Kalman filter, which is a variant of the PLF, this procedure, as tested in simulation, produced unbiased estimates for the bearings-only problem.[8]

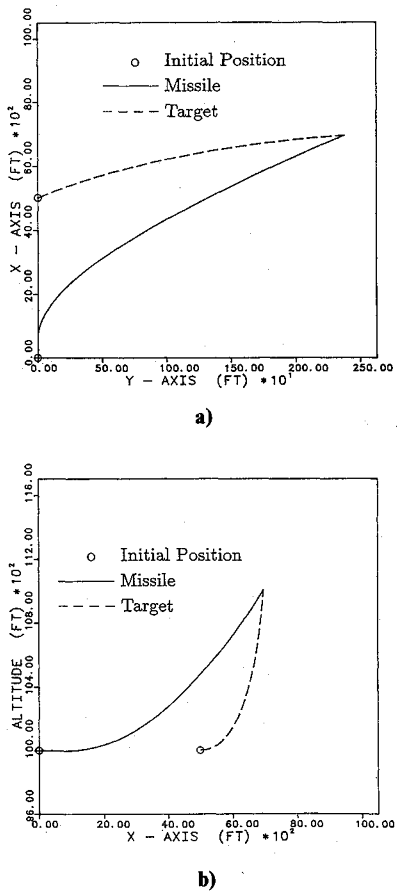


Fig. 2 LQGL trajectory in: a) $X_1Y_1$ plane; b) $X_1Z_1$ plane.

Table 1 Constants used in computations

| |
|---|
| Launch conditions: |
| Range = 5000 ft |
| Aspect angle = 30 deg |
| Off-boresight angle = 0.0 deg |
| Altitude = 10,000 ft |
| Initial error variance: |
| $P_{11} = P_{22} = P_{33} = 250{,}000$ ft$^2$ |
| $P_{44} = P_{55} = P_{66} = 10{,}000$ ft$^2$/s$^2$ |
| $P_{77} = P_{88} = P_{99} = 25{,}000$ ft$^2$/s$^4$ |
| Constants: |
| $c = 10{,}000$ s$^{-3}$ |
| $\lambda = 1$ s$^{-1}$ |
| $w = 0.17$ s$^{-4}$ |
| Sample period = 0.02 s |

In a matrix form, the dynamical model and the measurement model for the filter can be expressed as

$$\dot{X} = AX + BU \tag{28}$$

and

$$HX = 0 \tag{29}$$

Here, the state vector is given by

$$X = [x\ y\ z\ u\ v\ w\ a_{T_x}\ a_{T_y}\ a_{T_z}]^T \tag{30}$$

and the control vector is written as

$$U = [a_{M_x}\ a_{M_y}\ a_{M_z}]^T \tag{31}$$

Finally, the matrices $A$, $B$, and $H$ are defined as

$$A = \begin{bmatrix} 0_3 & I_3 & 0_3 \\ 0_3 & 0_3 & I_3 \\ 0_3 & 0_3 & -\lambda I_3 \end{bmatrix}, \quad B = \begin{bmatrix} 0_3 \\ -I_3 \\ 0_3 \end{bmatrix} \tag{32}$$

$$H = \begin{bmatrix} \sin\theta & -\cos\theta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{33}$$

where $0_3$ and $I_3$ are $3 \times 3$ zero and identity matrices, respectively.

The linear system represented by Eqs. (28) and (29) is observable[14] if the observability Gramian

$$I(t_f,t_0) = \int_{t_0}^{t_f} \Phi^T H^T H \Phi \, \mathrm{d}t \tag{34}$$

is nonsingular. In this equation, $\Phi = \Phi(t,t_0)$ is the transition matrix associated with $\dot{X} = AX$, i.e., it satisfies the differential equation $\dot{\Phi} = A\Phi$ and the boundary condition $\Phi(t_0,t_0) = I$. For $A$ defined by Eq. (32), the transition matrix is given by

$$\Phi(t,t_0) = \begin{bmatrix} I_3 & (\Delta t) I_3 & \dfrac{e^{-\lambda\Delta t} + \lambda\Delta t - 1}{\lambda^2} I_3 \\ 0_3 & I_3 & -\dfrac{e^{-\lambda\Delta t} - 1}{\lambda} I_3 \\ 0_3 & 0_3 & e^{-\lambda\Delta t} I_3 \end{bmatrix} \tag{35}$$

where $\Delta t = t - t_0$.

It is argued that the more nonsingular the observability matrix [Eq. (34)] is, the more observable the system is. However, to develop a maximum-observability trajectory, a scalar performance index is needed. Here, the performance index is taken to be the trace of the positive-definite matrix [Eq. (34)], i.e.,

$$J = \mathrm{tr} W \int_{t_0}^{t_f} \Phi^T H^T V H \Phi \, \mathrm{d}t \tag{36}$$

where $W$ and $V$ are positive-definite, diagonal weighting matrices. The constant matrix $W$ makes it possible to consider parts of the observability matrix and is written as

$$W = \begin{bmatrix} W_1 I_3 & 0_3 & 0_3 \\ 0_3 & W_2 I_3 & 0_3 \\ 0_3 & 0_3 & W_3 I_3 \end{bmatrix} \tag{37}$$

where $W_1$, $W_2$, and $W_3$ are constants. The time-varying matrix $V$ is included to weigh information enhancement toward the beginning of the trajectory; it is chosen to be

$$V = \begin{bmatrix} R^2 & 0 \\ 0 & R^2 \end{bmatrix} \tag{38}$$

where $R$ denotes the range-to-go and $R^2 = x^2 + y^2 + z^2$. In Eq. (36), the trace and the integral commute. Then, because of the property of the trace that $\mathrm{tr}(ABC) = \mathrm{tr}(CAB)$, the performance index can be rewritten as

$$J = \int_{t_0}^{t_f} \mathrm{tr} H^T V H \Phi W \Phi^T \, \mathrm{d}t \tag{39}$$

Next, Eqs. (24), (25), (33), (35), and (37–39) can be combined to yield

$$J = 2 \int_{t_0}^{t_f} \left[ W_1 + W_2 \Delta t^2 + \frac{W_3}{\lambda^4} (e^{-\lambda\Delta t} + \lambda\Delta t - 1)^2 \right] \times (x^2 + y^2 + z^2) \, \mathrm{d}t \tag{40}$$

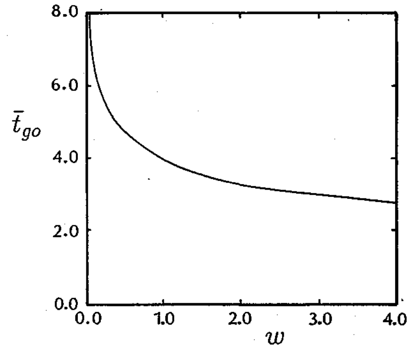


**Fig. 3 Conjugate point in DCGL.**

where $\Delta t = t - t_0$. Finally, if only position information is to be considered ($W_1 = 1$, $W_2 = 0$, $W_3 = 0$), a performance index for information enhancement becomes

$$\frac{J}{2} = \int_{t_0}^{t_f} (x^2 + y^2 + z^2) \, \mathrm{d}t \tag{41}$$

Since observability is to be maximized, Eq. (41) is added to Eq. (1) with a negative weight to form the dual-control performance index.

$$J = \frac{c}{2}(x_f^2 + y_f^2 + z_f^2) + \frac{1}{2}\int_{t_0}^{t_f} [(a_{M_x}^2 + a_{M_y}^2 + a_{M_z}^2) - w(x^2 + y^2 + z^2)] \, \mathrm{d}t \tag{42}$$

This performance index represents minimizing the miss distance with a penalty on the missile acceleration and with a "negative penalty" on the distance to the target. It has the additional benefit that the optimal control problem again splits into three separate problems—one along each axis. However, the optimal control problem now has the possibility of a conjugate point.

## IV. Dual-Control Guidance Law

The optimal control problem to be solved is that of finding the control $a_{M_y}(t)$, which minimizes the quadratic performance index

$$J = \frac{c}{2} y_f^2 + \frac{1}{2} \int_{t_0}^{t_f} (a_{M_y}^2 - w y^2) \, \mathrm{d}t \tag{43}$$

subject to the linear differential constraints [Eqs. (14–16)], the prescribed initial conditions [Eq. (17)], and the prescribed final condition [Eq. (12)].

This linear-quadratic problem leads to the optimal control or DCGL

$$a_{M_y} = C_1 y + C_2 v + C_3 a_{T_y} \tag{44}$$

where

$$C_1 = \Lambda_{21} Y_{11} + \Lambda_{22} Y_{21} + \Lambda_{23} Y_{31} \tag{45}$$

$$C_2 = \Lambda_{21} Y_{12} + \Lambda_{22} Y_{22} + \Lambda_{23} Y_{32} \tag{46}$$

$$C_3 = \Lambda_{21} Y_{13} + \Lambda_{22} Y_{23} + \Lambda_{23} Y_{33} \tag{47}$$

The matrices $\Lambda$ and $Y$ are developed in the Appendix. For $w = 0$, the DCGL reduces to the LQGL.

A difficulty associated with this guidance law is that the optimal path has a conjugate point at some time-to-go $\bar{t}_{go}$. Hence, it is important to choose the weighting parameter $w$ so that the conjugate point time-to-go is larger than the missile time-to-go $t_{go}$. The conjugate point occurs when the matrix $Y$ diverges. It has been determined numerically for a range of values of $w$ and is shown in Fig. 3.

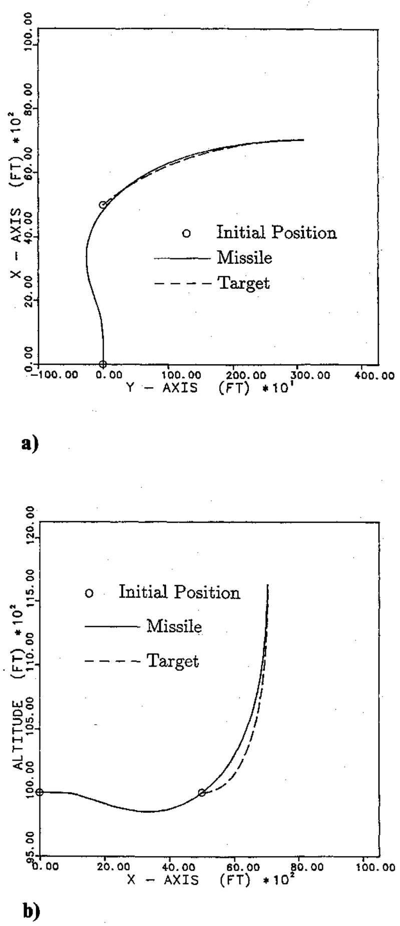


Fig. 4 DCGL trajectory in: a) $X_1Y_1$ plane; b) $X_1Z_1$ plane.

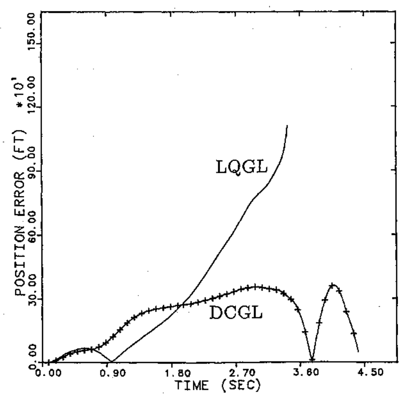


Fig. 5 Relative position tracking error.

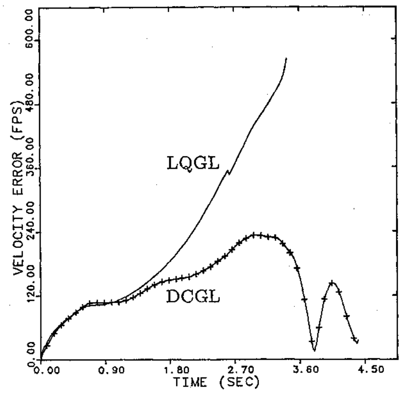


Fig. 6 Relative velocity tracking error.

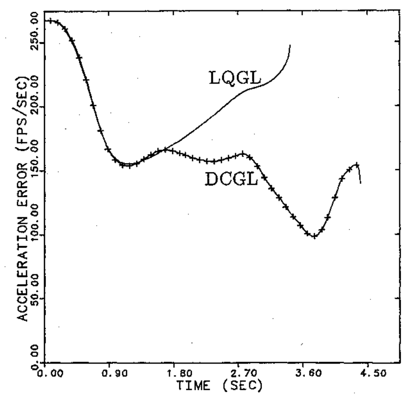


Fig. 7 Target acceleration tracking error.

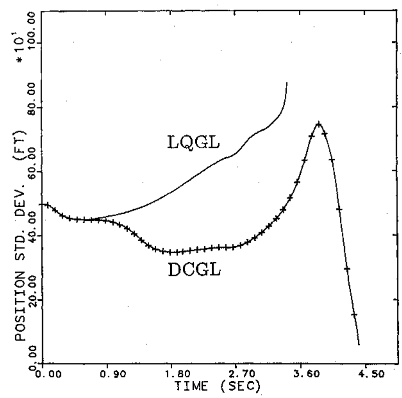


Fig. 8 Position standard deviation.

The next step is to determine a value for $w$ so that the miss distance is small, the missile acceleration is not excessive, and there is sufficient information content in the measurements. The values of $c$ and $\lambda$ are not changed. In order to allow some time for observability maneuvering, the time-to-go is increased by 10% before the gains are calculated.

If the time-to-go is calculated as range divided by closing speed, a reasonable value of $w$ is $0.17\ \mathrm{s}^{-4}$. The trajectory that results from the use of Eq. (39) as a guidance law with a sample time of 0.02 s is shown in Fig. 4. The final time is 4.40 s, and the miss distance is 2.88 ft. Note that the trajectory exhibits the character of the maximum information trajectories computed in Refs. 3 and 5, i.e., the missile swings in behind the target to maximize the line-of-sight angle rate.

Figure 4 shows that range divided by closing speed may not be a good way to calculate time-to-go for this guidance law. The angle between the missile velocity vector and the line of sight to the target can become large so that the closing speed becomes small and the predicted time-to-go becomes too large.

## V. Tracking Results

The miss distance for the DCGL is greater than the miss distance for the LQGL. This is to be expected because miss distance is being traded for observability. To show this, both guidance laws have been used to fly a bank-to-turn missile

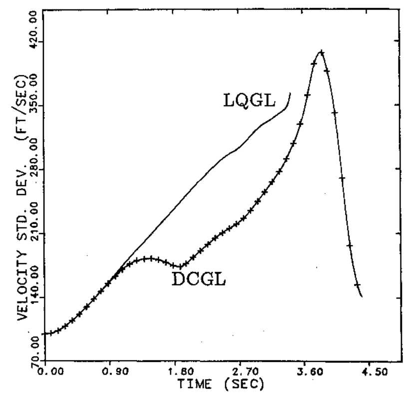


**Fig. 9 Velocity standard deviation.**

whose guidance system contains a navigation filter in cascade with the guidance law. However, the guidance law is fed the true states so that the filter does not influence the trajectory. The filter is used to obtain estimated states, which are then compared with the true states. Measurements are made at 0.02-s intervals and processed by the modified-gain extended Kalman filter being operated as an observer. The true relative position, the true relative velocity, and zero target acceleration are used for the initial state estimates. Initial values for the error covariance $P$ are presented in Table 1.

The tracking error presented is the magnitude of the error vector, i.e., the difference between the true vector and the estimated vector. Relative position error, relative velocity error, and target acceleration error are plotted in Figs. 5–7. Initial target acceleration error is due to a 9-$g$ target maneuver.

In general, measurements processed along the DCGL path provide better tracking results than those along the LQGL path. Since there is no process noise, the improved tracking results are due entirely to the shape of the trajectory along which the measurements are taken. Hence, the range term added to the performance index has the effect of improving the tracking performance of the observer.

Maximizing the observability is related directly to minimizing the error variance. Hence, as another indication of the usefulness of the DCGL, the maximum eigenvalues of the position part and the velocity part of the error variance are plotted in Figs. 8 and 9 in the form of a standard deviation (square root of the eigenvalue). It is noted that the standard deviations associated with the DCGL are less than those of the LQGL along the path and substantially less at the final time.

## VI. Discussion and Conclusions

The purpose of this paper has been to introduce a new linear-quadratic guidance law that has the effect of enhancing the information content of the flight path of a homing missile. Until now, it has not been possible to achieve a mechanizable guidance law that increases observability. The ability of this dual-control guidance law to improve state estimation has been demonstrated by comparing estimates obtained along its path with those obtained along a path generated by the original linear-quadratic guidance law.

Although the feasibility of this dual-control guidance law has been established, there is still much to be accomplished before it is useful. As an example, proper values for the penalty weights must be established to balance miss distance, missile acceleration, and filter enhancement for all engagement scenarios. Also, it seems reasonable that the value of $w$ should be adaptive, i.e., its value should be tied to some measure of the error variance. If the error variance becomes large, $w$ can be increased until the variance is reduced. Next, a current method for predicting time-to-go may not be adequate because this guidance law can cause the missile velocity vector to become perpendicular to the target velocity vector so that time-to-go predicted by range divided by closing speed becomes infinite. Hence, a new time-to-go algorithm is required. Finally, it has been suggested that the information weight $w$ be included in a form such as $w \exp[-a(t_f - t)]$, where $a$ is a constant, in order to delay the conjugate point. This feature is currently under investigation.

## Appendix

A general form of the linear-quadratic problem is the minimization of the quadratic performance index

$$J = \frac{1}{2} x_f^T T x_f + \frac{1}{2} \int_{t_0}^{t_f} (x^T U x + u^T V u)\, \mathrm{d}t \tag{A1}$$

subject to the linear differential constraints

$$\dot{x} = Mx + Nu \tag{A2}$$

subject to the prescribed initial conditions

$$t_0, x_0 \quad \text{given} \tag{A3}$$

and subject to the prescribed final condition

$$t_f \quad \text{given} \tag{A4}$$

Here, $x$ is an $n \times 1$ state vector; $u$ is an $m \times 1$ control vector, and dimensions $T$, $U$, $V$, $M$, and $N$ are $n \times n$, $n \times n$, $m \times m$, $n \times n$, and $n \times m$, respectively. The optimal control is given by[12]

$$u = Cx \tag{A5}$$

where the gain matrix $C$ is defined as

$$C = -V^{-1} N^T \Lambda X^{-1} \tag{A6}$$

The two $n \times n$ matrices $X$ and $\Lambda$ satisfy the differential equations

$$\dot{X} = MX - NV^{-1} N^T \Lambda \tag{A7}$$

$$\dot{\Lambda} = -UX - M^T \Lambda \tag{A8}$$

and the boundary conditions

$$X_f = I \tag{A9}$$

$$\Lambda_f = T \tag{A10}$$

To fit the dual-control problem into the linear-quadratic structure, it is seen that

$$x = [y\ v\ a_{T_y}]^T, \qquad u = [a_{M_y}] \tag{A11}$$

$$T = \begin{bmatrix} c & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} -w & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad V = [1] \tag{A12}$$

$$M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\lambda \end{bmatrix}, \quad N = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \tag{A13}$$

Then, the differential equations [Eqs. (A7) and (A8)] and their boundary conditions [Eqs. (A9) and (A10)] become

$$\dot{X}_{11} = X_{21}, \qquad X_{11f} = 1 \tag{A14}$$

$$\dot{X}_{12} = X_{22}, \qquad X_{12f} = 0 \tag{A15}$$

$$\dot{X}_{13} = X_{23}, \qquad X_{13f} = 0 \tag{A16}$$

$$\dot{X}_{21} = X_{31} - \Lambda_{21}, \qquad X_{21f} = 0 \tag{A17}$$

$$\dot{X}_{22} = X_{32} - \Lambda_{22}, \qquad X_{22f} = 1 \tag{A18}$$

$$\dot{X}_{23} = X_{33} - \Lambda_{23}, \qquad X_{23f} = 0 \tag{A19}$$

$$\dot{X}_{31} = -\lambda X_{31}, \qquad X_{31f} = 0 \tag{A20}$$

$$\dot{X}_{32} = -\lambda X_{32}, \qquad X_{32f} = 0 \tag{A21}$$

$$\dot{X}_{33} = -\lambda X_{33}, \qquad X_{33f} = 1 \tag{A22}$$

$$\dot{\Lambda}_{11} = wX_{11}, \qquad \Lambda_{11f} = c \tag{A23}$$

$$\dot{\Lambda}_{12} = wX_{12}, \qquad \Lambda_{12f} = 0 \tag{A24}$$

$$\dot{\Lambda}_{13} = wX_{13}, \qquad \Lambda_{13f} = 0 \tag{A25}$$

$$\dot{\Lambda}_{21} = -\Lambda_{11}, \qquad \Lambda_{21f} = 0 \tag{A26}$$

$$\dot{\Lambda}_{22} = -\Lambda_{12}, \qquad \Lambda_{22f} = 0 \tag{A27}$$

$$\dot{\Lambda}_{23} = -\Lambda_{13}, \qquad \Lambda_{23f} = 0 \tag{A28}$$

$$\dot{\Lambda}_{31} = -\Lambda_{21} + \lambda\Lambda_{31}, \qquad \Lambda_{31f} = 0 \tag{A29}$$

$$\dot{\Lambda}_{32} = -\Lambda_{22} + \lambda\Lambda_{32}, \qquad \Lambda_{32f} = 0 \tag{A30}$$

$$\dot{\Lambda}_{33} = -\Lambda_{23} + \lambda\Lambda_{33}, \qquad \Lambda_{33f} = 0 \tag{A31}$$

The solution process for these equations begins with the integration of Eqs. (A20–A22) to obtain

$$X_{31} = 0, \quad X_{32} = 0, \quad X_{33} = \exp[\lambda(t_f - t)] \tag{A32}$$

The general procedure for integrating the remaining equations is now discussed by considering the equations for $X_{13}$, $X_{23}$, $\Lambda_{13}$, and $\Lambda_{23}$. Combining Eqs. (A16), (A19), (A25), and (A28) gives

$$X_{13}^{(4)} - wX_{13} = \dot{X}_{33} \tag{A33}$$

Assuming a solution of the form

$$X_{13} = ae^{\alpha t} \tag{A34}$$

for the homogeneous part of Eq. (A33) leads to the following equation for $\alpha$:

$$\alpha^4 - w = 0 \tag{A35}$$

The four solutions of this equation are

$$\alpha = +k, \ -k, \ +ik, \ -ik \tag{A36}$$

where

$$k = w^{1/4} \tag{A37}$$

Hence, the general solution of Eq. (A33) has the form

$$X_{13} = a_1 \exp(kt) + a_2 \exp(-kt) + b_1 \cos kt + b_2 \sin kt \tag{A38}$$

where $a_1$, $a_2$, $b_1$, and $b_2$ are constants of integration. Because of the form of $X_{33}$, a particular solution for Eq. (A33) is given by

$$X_{13} = \frac{\lambda^2 X_{33}}{\lambda^4 - w} \tag{A39}$$

so that the complete solution of Eq. (A33) is the sum of Eqs. (A38) and (A39). Once $X_{13}$ is known, expressions for $X_{23}$, $\Lambda_{23}$, and $\Lambda_{13}$ follow from Eqs. (A16), (A19), and (A28). Finally, the constants of integration follow from the boundary conditions on these functions.

The solutions for all of the functions that are needed in the guidance law are given by the following equations, where $\tau = t_f - t$:

$$X_{11} = A_1 \exp(-k\tau) + A_2 \exp(k\tau) + \frac{1}{2}\cos k\tau + \frac{c}{2k^3}\sin kt \tag{A40}$$

$$X_{12} = \frac{1}{4k}\exp(-k\tau) - \frac{1}{4k}\exp(k\tau) - \frac{1}{2k}\sin k\tau \tag{A41}$$

$$X_{13} = \bar{A}_1 \exp(-k\tau) + \bar{A}_2 \exp(k\tau) + \bar{B}_1 k \exp(-k\tau) + \bar{B}_1 \lambda \sin k\tau + \bar{C}_1 \exp(\lambda\tau) \tag{A42}$$

$$X_{21} = A_1 k \exp(-k\tau) - A_2 k \exp(k\tau) - \frac{c}{2k^2}\cos k\tau + \frac{k}{2}\sin k\tau \tag{A43}$$

$$X_{22} = \frac{1}{4}\exp(-k\tau) + \frac{1}{4}\exp(k\tau) + \frac{1}{2}\cos k\tau \tag{A44}$$

$$X_{23} = \bar{A}_1 k \exp(-k\tau) - \bar{A}_2 k \exp(k\tau) - \bar{B}_1 k\lambda \cos k\tau + \bar{B}_1 k^2 \sin k\tau - \bar{C}_1 \lambda \exp(\lambda\tau) \tag{A45}$$

$$X_{31} = 0, \qquad X_{32} = 0, \qquad X_{33} = \exp(\lambda\tau) \tag{A46}$$

$$\Lambda_{21} = -A_1 k^2 \exp(-k\tau) - A_2 k^2 \exp(k\tau) + \frac{k^2}{2}\cos k\tau + \frac{c}{2k}\sin k\tau \tag{A47}$$

$$\Lambda_{22} = -\frac{k}{4}\exp(-k\tau) + \frac{k}{4}\exp(k\tau) - \frac{k}{2}\sin k\tau \tag{A48}$$

$$\Lambda_{23} = -\bar{A}_1 k^2 \exp(-k\tau) - \bar{A}_2 k^2 \exp(k\tau) + \bar{B}_1 k^3 \cos k\tau + \bar{B}_1 k^2 \lambda \sin k\tau - \bar{C}_1 \frac{w}{\lambda^2}\exp(\lambda\tau) \tag{A49}$$

In these equations,

$$A_1 = \frac{1}{4}\left(1 + \frac{c}{k^3}\right) \tag{A50}$$

$$A_2 = \frac{1}{4}\left(1 - \frac{c}{k^3}\right) \tag{A51}$$

$$\bar{A}_1 = \frac{(k^2\lambda^2 + w)(\lambda - k)}{4k^3(\lambda^4 - w)} \tag{A52}$$

$$\bar{A}_2 = -\frac{(k^2\lambda^2 + w)(\lambda + k)}{4k^3(\lambda^4 - w)} \tag{A53}$$

$$\bar{B}_1 = \frac{w - k^2\lambda^2}{2k^3(\lambda^4 - w)} \tag{A54}$$

$$\bar{C}_1 = \frac{\lambda^2}{\lambda^4 - w} \tag{A55}$$

This guidance law is presented in the text, where the matrix $Y$ is the inverse of $X$.

## Acknowledgments

This research was supported by the Air Force Office of Scientific Research under Grant AFOSR-84-0371 and by the Air Force Armament Laboratory under Contract F08635-87-K-0417.


## References


[1]Witsenhausen, H. S., "Separation of Estimation and Control of Discrete Time Systems," *Proceedings of the IEEE*, Vol. 59, Inst. of Electrical and Electronics Engineers, New York, 1971, pp. 1557–1566.

[2]Larson, S. W., "Maximum Information Trajectory for an Air-to-Air Missile Intercept," M.S. Thesis, Univ. of Texas, Austin, TX, Dec. 1980.

[3]Speyer, J. L., Hull, D. G., Tseng, C. -Y., and Larson, S. W., "Estimation Enhancement by Trajectory Modulation for Homing Missiles," *Journal of Guidance, Control, and Dynamics*, Vol. 7, March–April, 1984, pp. 167–174.

[4]Tseng, C. -Y., "A Study of Maximum Information Trajectories for Homing Missile Guidance," Ph.D. Dissertation, Univ. of Texas, Austin, TX, Dec. 1983.

[5]Hull, D. G., Speyer, J. L., and Tseng, C. -Y., "Maximum-Information Guidance for Homing Missiles," *Journal of Guidance, Control, and Dynamics*, Vol. 8, July–Aug., 1985, pp. 494–497.

[6]Bernard, W. P. H., "Filter Enhancement for Air-to-Air Missiles with Angle-Only Measurements," M.S. Thesis, Univ. of Texas, Austin, TX, Dec. 1985.

[7]Aidala, V. J. and Nardone, S. C., "Biased Estimation Properties of the Pseudolinear Tracking Filter," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. AES-18, July 1982, pp. 432–441.

[8]Song, T. L. and Speyer, J. L., "A Stochastic Analysis of a Modified-Gain Extended Kalman Filter with Applications to Estimation with Bearings-Only Measurements," *IEEE Transactions on Automatic Control*, Vol. AC-30, Oct. 1985, pp. 897–904.

[9]Speyer, J. L., Hull, D. G., and Bernard, W. P. H., "Performance of the Modified-Gain Extended Kalman Filter Along an Enhanced Information Path of a Homing Missile," *Proceedings of the AIAA Guidance and Control Conference*, AIAA, New York, Aug. 1986.

[10]Vergez, P. L., "Closed-Loop System Analysis Using a Lyapunov Stability Theory," Ph.D. Dissertation, Univ. of Texas, Austin, TX, May 1987.

[11]Burris, D. B., "A Dual-Control Guidance Algorithm for Air-to-Air Homing Missiles with Bearings-Only Measurements," M.S. Thesis, Univ. of Texas, Austin, TX, in preparation.

[12]Bryson, A. E. and Ho, Y.-C., *Applied Optimal Control*, Hemisphere, New York, 1975.

[13]Speyer, J. L. and Hull, D. G., "Comparison of Several Extended Kalman Filter Formulations for Homing Missile Guidance," *Proceedings of the AIAA Guidance and Control Conference*, AIAA, New York, Aug. 1980, pp. 392–398.

[14]Chen, C.-T., *Linear System Theory and Design*, Holt, Rinehart and Winston, New York, 1984.